

L -PACKETS AND FORMAL DEGREES FOR $\mathrm{SL}_2(K)$ WITH K A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2

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ABSTRACT. Let $\mathcal{G} = \mathrm{SL}_2(K)$ with K a local function field of characteristic 2. We review Artin-Schreier theory for the field K , and show that this leads to a parametrization of L -packets in the smooth dual of \mathcal{G} . We relate this to a recent geometric conjecture. The L -packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L -packets by biquadratic extensions. We compute the formal degrees of the elements in the supercuspidal packets.

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1. INTRODUCTION

In this article we consider a local function field K of characteristic 2, namely $K = \mathbb{F}_q((\varpi))$, the field of Laurent series with coefficients in \mathbb{F}_q , with $q = 2^f$. This example is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((\varpi))$.

We consider $\mathcal{G} = \mathrm{SL}_2(K)$. Drawing on the accounts in [5, 16, 17], we review Artin-Schreier theory, adapted to the local function field $\mathbb{F}_q((\varpi))$. This leads to a parametrization of L -packets in the smooth dual of \mathcal{G} . In this article, we reserve the term L -packets for the ones which are not singletons.

The L -packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L -packets by biquadratic extensions. There are countably many supercuspidal packets.

By *canonical formal degree* we shall mean formal degree with respect to the Euler-Poincaré measure on \mathcal{G} , as in [12]. We compute the canonical formal degrees of the elements in the supercuspidal packets, relying on the Formal Degree component of the local Langlands correspondence, see [12, §6]. The canonical formal degrees are all dyadic rationals, in fact they are integer powers of 2. They depend on the residue degree f , and on the breaks in the lower ramification filtration of the Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The commutative triangle in Theorem 4.3, and the bijective maps 5.2, 5.3, 5.4 in §5, amount to a proof, for \mathcal{G} , of the *tempered* version of the geometric conjecture in [1].

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2. ARTIN-SCHREIER THEORY

Let K be a local field with positive characteristic p . The cyclic extensions of K whose degree n is coprime with p are described by Kummer theory. It is well known that any cyclic extension L/K of degree n , $(n, p) = 1$, is generated by a root α of an irreducible polynomial $x^n - a \in K[x]$. If $\alpha \in K^s$ is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree n and is called a Kummer extension of K .

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by $ch(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^n - x - a$. Essentially, every cyclic extension of K with degree $p = ch(K)$ is generated by a root α of $x^p - x - a \in K[x]$.

We fix an algebraic closure \overline{K} of K and a separable closure K^s of K in \overline{K} . Let \wp denote the Artin-Schreier endomorphism of the additive group K^s [9]:

$$\wp : K^s \rightarrow K^s, \quad x \mapsto x^p - x.$$

Given $a \in K$ denote by $K(\wp^{-1}(a))$ the extension $K(\alpha)$, where $\wp(\alpha) = a$ and $\alpha \in K^s$. We have the following characterization of finite cyclic Artin-Schreier extensions of degree p :

Theorem 2.1. (i) *Given $a \in K$, either $\wp(x) - a \in K[x]$ has one root in K in which case it has all the p roots are in K , or is irreducible.*
(ii) *If $\wp(x) - a \in K[x]$ is irreducible then $K(\wp^{-1}(a))/K$ is a cyclic extension of degree p , with $\wp^{-1}(a) \subset K^s$.*

(iii) If L/K be a finite cyclic extension of degree p , then $L = K(\wp^{-1}(a))$, for some $a \in K$.

(See [16, p.34] for more details.)

We fix now some notation. K is a local field with characteristic $p > 1$ with finite residue field k . The field of constants $k = \mathbb{F}_q$ is a finite extension of \mathbb{F}_p , with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$.

Let \mathfrak{o} be the ring of integers in K and denote by $\mathfrak{p} \subset \mathfrak{o}$ the (unique) maximal ideal of \mathfrak{o} . This ideal is principal and any generator of \mathfrak{p} is called a uniformizer. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi\mathfrak{o} \cong \varpi\mathbb{F}_q[[\varpi]]$.

A normalized valuation on K will be denoted by ν , so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$. The group of units is denoted by \mathfrak{o}^\times .

2.1. The Artin-Schreier symbol. Let L/K be a finite Galois extension. Let $N_{L/K}$ be the norm map and denote $G_{L/K}^{ab} = \text{Gal}(L/K)^{ab}$ the abelianization of $\text{Gal}(L/K)$. The reciprocity map is a group isomorphism

$$(2.1) \quad K^\times / N_{L/K} L^\times \xrightarrow{\cong} G_{L/K}^{ab}.$$

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism $K^\times \rightarrow K^\times / N_{L/K} L^\times$

$$(2.2) \quad b \in K^\times \mapsto (b, L/K) \in G_{L/K}^{ab}.$$

From the Artin symbol we obtain a pairing

$$(2.3) \quad K \times K^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha,$$

where $\wp(\alpha) = a$, $\alpha \in K^s$ and $L = K(\alpha)$.

Definition 2.2. Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is defined to be

$$[a, b] = (b, L/K)(\alpha) - \alpha.$$

We summarize some important properties of the Artin-Schreier symbol.

Proposition 2.3. The Artin-Schreier symbol is a bilinear map satisfying the following properties:

- (i) $[a_1 + a_2, b] = [a_1, b] + [a_2, b];$
- (ii) $[a, b_1 b_2] = [a, b_1] + [a, b_2];$
- (iii) $[a, b] = 0, \forall a \in K \Leftrightarrow b \in N_{L/K} L^\times, L = K(\alpha) \text{ and } \wp(\alpha) = a;$
- (iv) $[a, b] = 0, \forall b \in K^\times \Leftrightarrow a \in \wp(K).$

(See [9, p.341])

2.2. The groups $K/\wp(K)$ and $K^\times/K^{\times p}$. In this section we recall some properties of the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing (2.3).

Consider the additive group K . The index of $\wp(K)$ in K is infinite [6, p.146]. Hence, $K/\wp(K)$ is infinite.

Proposition 2.4. *$K/\wp(K)$ is a discrete abelian torsion group.*

Proof. The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}).$$

The restriction $\wp : \mathfrak{p} \rightarrow \mathfrak{p}$ is an isomorphism, see [5, Lemma 8]. Hence,

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of K and $K/\wp(K)$ is discrete. Since $\wp(K)$ is annihilated by p , $K/\wp(K)$ is a torsion group. \square

Now we concentrate on the multiplicative group K^\times . The subgroup $K^{\times p}$ is not open in K^\times and the index $[K^\times : K^{\times p}]$ is infinite [6, Lemma p.115]. Hence, $K^\times/K^{\times p}$ is infinite. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

Proposition 2.5. *$K^\times/K^{\times p}$ is a profinite abelian p -torsion group.*

Proof. There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$. By [8, p.25], the group of units \mathfrak{o}^\times is a direct product of countable many copies of the ring of p -adic integers

$$\mathfrak{o}^\times \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \dots = \prod_{\mathbb{N}} \mathbb{Z}_p.$$

Give \mathbb{Z} the discrete topology and \mathbb{Z}_p the p -adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

$K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \dots = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$

Denote by $y = \prod_n y_n$ and element of $\prod_{\mathbb{N}} \mathbb{Z}_p$, where $y_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$, for every n .

The map

$$\varphi : \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \rightarrow \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}, (x, y) \mapsto (x \pmod{p}, \prod_n pr_0(y_n))$$

where $pr_0(y_n) = a_{0,n}$ is the projection, is clearly a group homomorphism.

Now, $\mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a topological group for the product topology, where each component $\mathbb{Z}/p\mathbb{Z}$ has the discrete topology. Moreover, it is compact by Tyconoff Theorem, Hausdorff and totally disconnected [2, TGI.84, Prop. 10]. Therefore, $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a profinite group.

Since

$$\ker \varphi = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

it follows that there is an isomorphism of topological groups

$$K^{\times}/K^{\times p} \cong \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

where $K^{\times}/K^{\times p}$ is given the quotient topology. Therefore, $K^{\times}/K^{\times p}$ is profinite. \square

From Propositions 6.1 and 2.5, $K/\wp(K)$ is a discrete abelian group and $K/K^{\times p}$ is an abelian profinite group, both annihilated by $p = \text{ch}(K)$. Therefore, Pontryagin duality coincides with $\text{Hom}(-, \mathbb{Z}/p\mathbb{Z})$ on both of these groups, see [17]. See also [13] for more details on Pontryagin duality. The pairing (2.3) restricts to a pairing

$$(2.1) \quad [., .] : K/\wp(K) \times K^{\times}/K^{\times p} \rightarrow \mathbb{Z}/p\mathbb{Z}.$$

which we refer from now on to the **Artin-Schreier pairing**. It follows from (iii) and (iv) of Proposition 2.3, the pairing is nondegenerate (see also [17, Proposition 3.1]). The next result shows that the pairing is perfect.

Proposition 2.6. *The Artin-Schreier symbol induces isomorphisms of topological groups*

$$K^{\times}/K^{\times p} \xrightarrow{\cong} \text{Hom}(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b])$$

and

$$K/\wp(K) \xrightarrow{\cong} \text{Hom}(K^{\times}/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b])$$

Proof. The result follows by taking $n = 1$ in Proposition 5.1 of [17], and from the fact that Pontryagin duality for the groups $K/\wp(K)$ and $K^{\times}/K^{\times p}$ coincide with $\text{Hom}(-, \mathbb{Z}/p\mathbb{Z})$ duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. \square

Let B be a subgroup of the additive group of K with finite index such that $\wp(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent p is again a finite abelian Galois extension of exponent p . Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent p . On the other hand, if L/K is a finite abelian Galois extension of exponent p , then $L = K_B$ for some subgroup $\wp(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent p , which we denote by $K_p = K(\wp^{-1}(K))$. The extension K_p/K is infinite and Galois. The corresponding Galois group $G_p = \text{Gal}(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^{\times p}$, see [17, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^{\times 2}$ and will play a fundamental role in the sequel.

3. QUADRATIC CHARACTERS

From now on we take K to be a local function field with $ch(K) = 2$. Therefore, K is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

Recall that a character of K^\times is a group homomorphism

$$\chi : K^\times \rightarrow \mathbb{T}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Denote by $\widehat{K^\times}$ the group of characters of K^\times . There is a canonical isomorphism

$$\widehat{K^\times} \cong \widehat{\mathbb{Z} \times \mathfrak{o}^\times} \cong \mathbb{T} \times \widehat{\mathfrak{o}^\times}.$$

Therefore, given a character $\chi \in \widehat{K^\times}$, we may write $\chi = z^\nu \chi_0$, where $z \in \mathbb{T}$, ν is the valuation and $\chi_0 \in \widehat{\mathfrak{o}^\times}$. If $\chi_0 \equiv 1$ we say that χ is unramified. A character χ of K^\times is called quadratic if $\chi^2 = 1$. Since the unique quadratic character of \mathbb{Z} is $(n \mapsto (-1)^n)$, a nontrivial quadratic character has the form $\chi = (-1)^\nu \chi_0$, with $\chi_0^2 = 1$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [8, p.25],

$$\mathfrak{o}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots = \prod_{\mathbb{N}} \mathbb{Z}_2$$

with countably infinite many copies of \mathbb{Z}_2 , the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By Proposition 2.5, there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group

$$\mathbb{F}_q((\varpi))^\times / \mathbb{F}_q((\varpi))^{\times 2} \cong \prod_{\mathbb{N}} 2\mathbb{Z}_2 = G_2$$

where G_2 is the Galois group of the maximal abelian extension of exponent 2. Since G_2 is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol

$$[a, \cdot) : K^\times / K^{\times 2} \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Now, let φ denote the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character $\chi_a = \varphi([a, \cdot])$:

$$(3.1) \quad \chi_a : K^\times \rightarrow \mathbb{C}^\times.$$

Proposition 2.6 shows that every quadratic character of $\mathbb{F}_q((\varpi))^\times$ arises in this way.

Example 3.1. *The unramified quadratic extension of K is $K(\wp^{-1}(\mathfrak{o}))$, see [5] proposition 12. According to Dalawat, the group $K/\wp(K)$ may be regarded as an \mathbb{F}_2 -space and the image of \mathfrak{o} under the canonical surjection $K \rightarrow K/\wp(K)$ is an \mathbb{F}_2 -line, i.e., isomorphic to \mathbb{F}_2 . Since $\wp|_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathfrak{p}$ is an isomorphism, the image of \mathfrak{p} in $K/\wp(K)$ is $\{0\}$, see lemma 8 in [5]. Now, choose any $a_0 \in \mathfrak{o} \setminus \mathfrak{p}$. The quadratic character $\chi_{a_0} = \varphi([a_0, \cdot])$ associated with $K(\wp^{-1}(\mathfrak{o}))$ via class field theory is precisely the unramified character ($n \mapsto (-1)^n$) from above. Note that any other choice $b_0 \in \mathfrak{o} \setminus \mathfrak{p}$, with $a_0 \neq b_0$, gives the same unique unramified character, since there is only one nontrivial coset $a_0 + \wp(K)$ for $a_0 \in \mathfrak{o} \setminus \mathfrak{p}$.*

Let \mathcal{G} denote $\mathrm{SL}_2(K)$, let \mathcal{B} be the standard Borel subgroup of \mathcal{G} , let \mathcal{T} be the diagonal subgroup of \mathcal{G} . Let χ be a character of \mathcal{T} . Then, χ inflates to a character of \mathcal{B} . Denote by π_χ the (unitarily) induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$. The representation space of V_χ of π_χ consists of locally constant complex valued functions $f : \mathcal{G} \rightarrow \mathbb{C}$ such that, for every $a \in K^\times$, $b \in K$ and $g \in \mathcal{G}$, we have

$$f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g\right) = |a| \chi(a) f(g)$$

The action of \mathcal{G} on V_χ is by right translation. The representations (π_χ, V_χ) are called (unitary) principal series of $\mathcal{G} = \mathrm{SL}_2(K)$.

Let χ be a quadratic character of K^\times . The reducibility of the induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$ is well known in zero characteristic. W. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic p .

Theorem 3.2. [3, 4] *The representation $\pi_\chi = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$ is reducible if, and only if, χ is either $|\cdot|^\pm$ or a nontrivial quadratic character of K^\times .*

For a proof see [3, Theorems 1.7, 1.9] and [4, §9].

From now on, χ will be a quadratic character. It is a classical result that the unitary principal series for GL_2 are irreducible. For a representation of GL_2 parabolically induced by $1 \otimes \chi$, Clifford theory tells

us that the dimension of the intertwining algebra of its restriction to SL_2 is 2. This is exactly the induced representation of SL_2 by χ :

$$\mathrm{Ind}_{\tilde{B}}^{\mathrm{GL}_2(K)}(1 \otimes \chi)_{|\mathrm{SL}(2,K)} \xrightarrow{\simeq} \mathrm{Ind}_B^{\mathrm{SL}_2(K)}(\chi)$$

where \tilde{B} denotes the standard Borel subgroup of $\mathrm{GL}_2(K)$. This leads to reducibility of the induced representation $\mathrm{Ind}_B^G(\chi)$ into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

$$(3.2) \quad \pi_\chi = \mathrm{Ind}_B^G(\chi) = \pi_\chi^+ \oplus \pi_\chi^-$$

define an L -packet $\{\pi_\chi^+, \pi_\chi^-\}$ for SL_2 .

4. A COMMUTATIVE TRIANGLE

In this section we confirm part of the geometric conjecture in [1] for $\mathrm{SL}_2(\mathbb{F}_q((\varpi)))$. We begin by recalling the underlying ideas of the conjecture.

Let \mathcal{G} be the group of K -points of a connected reductive group over a nonarchimedean local field K . We have the *Bernstein decomposition*

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^\mathfrak{s}$$

over all points $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ the Bernstein spectrum of \mathcal{G} , see [14].

Let $\chi_{a_0} = \varphi([a_0, \cdot])$ denote the unramified character of K^\times associated with the unramified quadratic extension $K(\alpha_0) = K(\wp^{-1}(\mathfrak{o}))$ as in example 3.1. Fix a quadratic character $\chi_a = \varphi([a, \cdot])$ associated via class field theory with the quadratic extension $K(\alpha)$ (in a fixed algebraic closure \overline{K}), where $\alpha^2 - \alpha = a$.

Proposition 4.1. *There is a unique quadratic extension $K(\beta)$ with associated character χ_{a_0+a} . Moreover, $\chi_{a_0+a} = \chi_{a_0}\chi_a$.*

Proof. The compositum $K(\alpha)K(\alpha_0)$ is Galoisian over K , with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, contains three subfields, which are quadratic extensions of K , namely $K(\alpha_0)$, $K(\alpha)$ and, say, $K(\beta)$. The extension $K(\beta)$ is such that $\beta^2 - \beta = a_0 + a$, and has an associated quadratic character given by χ_{a_0+a} . Hence

$$\chi_{a_0+a} = \varphi([a_0 + a, \cdot]) = \varphi([a_0, \cdot] + [a, \cdot]) = \varphi([a_0, \cdot])\varphi([a, \cdot]) = \chi_{a_0}\chi_a.$$

□

By theorem 3.2, the induced representations

$$\pi_{a_0} = \mathrm{Ind}_B^G(\chi_{a_0}), \pi_a = \mathrm{Ind}_B^G(\chi_a) \text{ and } \pi_{a_0+a} = \mathrm{Ind}_B^G(\chi_{a_0+a})$$

are reducible and split into a direct sum of two irreducible component.

Central to the geometric conjecture is the concept of extended quotient of the second kind, which we now define.

Let W be a finite group and let X be a complex affine algebraic variety. Suppose that W is acting on X as automorphisms of X . Define

$$\tilde{X}_2 := \{(x, \tau) : \tau \in \mathbf{Irr}(W_x)\}.$$

Then W acts on \tilde{X}_2 :

$$\alpha(x, \tau) = (\alpha \cdot x, \alpha_* \tau).$$

Definition 4.2. *The extended quotient of the second kind is defined as*

$$(X//W)_2 := \tilde{X}_2/W.$$

Thus the extended quotient of the second kind is the ordinary quotient for the action of W on \tilde{X}_2 .

Theorem 4.3. *Let $\mathcal{G} = \mathrm{SL}_2(K)$ with $K = \mathbb{F}_q((\varpi))$. Let $\mathfrak{s} = [\mathcal{T}, \chi]_G$ be a point in the Bernstein spectrum for the principal series of \mathcal{G} . Let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be the corresponding Bernstein component in $\mathbf{Irr}(\mathcal{G})$. Then the conjecture [1] is valid for $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ i.e. there is a commutative triangle of natural bijections*

$$\begin{array}{ccc} & (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 & \\ \swarrow & & \searrow \\ \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & \xrightarrow{\quad} & \mathfrak{L}(G)^{\mathfrak{s}} \end{array}$$

where $\mathfrak{L}(G)^{\mathfrak{s}}$ denotes the equivalence classes of enhanced parameters attached to \mathfrak{s} .

Proof. We recall that (G, T) are the complex dual groups of $(\mathcal{G}, \mathcal{T})$. Let \mathbf{W}_K denote the Weil group of K . If φ is an L -parameter

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G$$

then \mathcal{S}_{φ} is defined as follows:

$$\mathcal{S}_{\varphi} := \pi_0 C_G(\mathrm{im} \varphi).$$

By an *enhanced* Langlands parameter, we shall mean a pair (φ, ρ) where φ is a parameter and $\rho \in \mathbf{Irr}(\mathcal{S}_{\varphi})$. Following Reeder [12], we shall denote an enhanced Langlands parameter by $\varphi(\rho)$.

Case 1. Let χ be a quadratic character of \mathcal{T} : $\chi^2 = 1, \chi \neq 1$. Let L/K be the quadratic extension determined by χ . Now G contains a unique (up to conjugacy) subgroup $H \simeq \mathbb{Z}/2\mathbb{Z}$. Each quadratic extension L/K creates a parameter

$$\varphi_L : \mathbf{W}_K \rightarrow \mathrm{Gal}(L/K) \rightarrow G.$$

The map $\mathrm{Gal}(L/K) \rightarrow H$ factors through $K^{\times}/N_{L/K}L^{\times}$:

$$\varphi_L : \mathbf{W}_K \rightarrow \mathrm{Gal}(L/K) \simeq K^{\times}/N_{L/K}L^{\times} \rightarrow H \rightarrow G.$$

which shows that φ_L is the parameter attached to the packet π_{χ} .

To compute \mathcal{S}_{φ_L} , let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

$$C_G(\text{im } \varphi_L) = T \sqcup wT$$

So φ is a non-discrete parameter, and we have

$$\mathcal{S}_{\varphi_L} \simeq \mathbb{Z}/2\mathbb{Z}.$$

We have two enhanced Langlands parameters, namely $\varphi_L(\text{triv})$ and $\varphi_L(\rho)$ where ρ is the nontrivial character of \mathcal{S}_{φ_L} .

Now define

$$\chi(\varpi) = \chi \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$$

where ϖ is a uniformizer in K .

Since $\chi^2 = 1$, there is a point of reducibility. We have, at the level of elements,

$$\begin{array}{ccc} & \{(\chi(\varpi), \text{triv}), (\chi(\varpi), \rho)\} & \\ \swarrow & & \searrow \\ \{\pi_{\chi}^+, \pi_{\chi}^-\} & \xrightarrow{\hspace{10em}} & \{\varphi_{\chi}(\text{triv}), \varphi_{\chi}(\rho)\} \end{array}$$

Case 2. Let $\chi = 1$. The *principal parameter* is the composite map

$$\varphi_0 : \mathbf{W}_K \times \text{SL}_2(\mathbb{C}) \rightarrow \text{PSL}(2, \mathbb{C}).$$

defined by extending the *principal* homomorphism $\text{SL}_2(\mathbb{C}) \rightarrow \text{PSL}_2(\mathbb{C})$ trivially on \mathbf{W}_K , is a canonical discrete parameter for which $\mathcal{S}_{\varphi_0} = 1$. In the local Langlands correspondence for \mathcal{G} , the enhanced parameter $\varphi_0(\text{triv})$ corresponds to the Steinberg representation $\text{St}_{\mathcal{G}}$, see [12, 6.1.8].

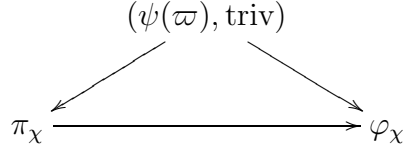
Let φ_1 be the unique parameter for which $\varphi_1(\mathbf{W}_K \times \text{SL}_2(\mathbb{C})) = 1$. We have

$$\text{im } \varphi_1 = 1, \quad C_G(\text{im } \varphi_1) = G, \quad \mathcal{S}_{\varphi_1} = 1.$$

There is a unique enhanced parameter, namely $\varphi_1(\text{triv})$. We have, at the level of elements, the commutative triangle

$$\begin{array}{ccc} & \{(1, \text{triv}), (1, \rho)\} & \\ \swarrow & & \searrow \\ \{\text{St}_{\mathcal{G}}, 1_{\mathcal{G}}\} & \xrightarrow{\hspace{10em}} & \{\varphi_0(\text{triv}), \varphi_1(\text{triv})\} \end{array}$$

Case 3. $\chi^2 \neq 1$. There are no points of reducibility, and we have a commutative triangle of sets, each with one element:



□

Corollary 4.4. *Let L/K be a quadratic extension of K . The L -parameters φ_L serve as parameters for the L -packets in the principal series of $\text{SL}_2(K)$.*

It follows from §3 that there are countably many L -packets in the principal series of $\text{SL}_2(K)$.

5. THE TEMPERED DUAL

The following picture



serves two purposes. First, it is an accurate portrayal of the extended quotient of the second kind

$$(\mathbb{T}/W)_2$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the generator of $W = \mathbb{Z}/2\mathbb{Z}$ acts on \mathbb{T} sending z to z^{-1} . Secondly, it is (conjecturally) an accurate portrayal of a connected component in the tempered dual of \mathcal{G} .

The topology on $(\mathbb{T}/W)_2$ comes about as follows. Let

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W)$$

denote the primitive ideal space of the noncommutative C^* -algebra $C(\mathbb{T}) \rtimes W$. By the classical Mackey theory for semidirect products, we have a canonical bijection

$$(5.1) \quad \mathbf{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}/W)_2.$$

The primitive ideal space on the left-hand side of (5.1) admits the Jacobson topology. So the right-hand side of (5.1) acquires, by transport of structure, a compact non-Hausdorff topology. The picture at the beginning of this section is intended to portray this topology. We shall see that the Langlands parameters respect this topology. The double-points in the picture arise precisely when the corresponding induced representation has length 2.

The Plancherel Theorem of Harish-Chandra is valid for any local non-archimedean field, see Waldspurger [18]. This implies that, in the case at hand, the discrete series and the unitary principal series enter into the Plancherel formula. That is, the tempered dual of \mathcal{G}

comprises the discrete series and the irreducible constituents in the unitary principal series.

We now focus on the case of induced elements.

Suppose $\chi^2 \neq 1$, with $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$. Let ψ be an unramified unitary character of \mathcal{T} . Then we have a natural bijection

$$(5.2) \quad \mathbf{Irr}^{\text{temp}}(G)^{\mathfrak{s}} \simeq \mathbb{T}, \quad \text{Ind } \pi_{\psi\chi} \mapsto \psi(\varpi).$$

Suppose $\chi^2 = 1, \chi \neq 1$, with $\mathfrak{s} = [T, \chi]_G$. Let $W = \mathbb{Z}/2\mathbb{Z}$. Then we have a bijective map

$$(5.3) \quad \mathbf{Irr}^{\text{temp}}(G)^{\mathfrak{s}} \simeq (\mathbb{T} // W)_2.$$

This map is defined as follows. Let ρ denote the nontrivial character of W .

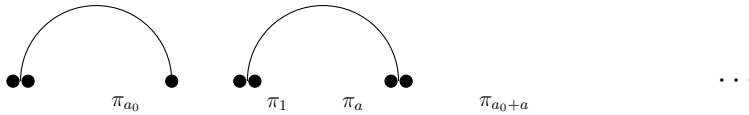
- If $\psi^2 \neq 1$, send $\text{Ind } \pi_{\psi\chi}$ to $\psi(\varpi)$.
- If $\psi = 1$, send the pair of irreducible constituents $\pi_{\chi}^+, \pi_{\chi}^-$ to the pair of points $(1, \text{triv}), (1, \rho) \in (T // W)_2$.
- If $\psi = \epsilon$ the unique unramified *quadratic* character of \mathcal{T} , send the pair of irreducible constituents $\pi_{\epsilon\chi}^+, \pi_{\epsilon\chi}^-$ to the pair of points $(-1, \text{triv}), (-1, \rho) \in (T // W)_2$.

Suppose $\chi = 1$ and let $\mathfrak{s}_0 = [T, 1]_G$. Then we have a continuous bijection which is *not* a homeomorphism:

$$(5.4) \quad \mathbf{Irr}^{\text{temp}}(G)^{\mathfrak{s}} \rightarrow (\mathbb{T} // W)_2.$$

- If $\psi^2 \neq 1$, send $\text{Ind } \pi_{\psi\chi}$ to $\psi(\varpi)$.
- If $\psi = 1$, send the irreducible representations $\text{triv}_{\mathcal{G}}, \text{St}_{\mathcal{G}}$ to the pair of points $(1, \text{triv}), (1, \rho) \in (T // W)_2$.
- If $\psi = \epsilon$ send the pair of irreducible constituents $\pi_{\epsilon}^+, \pi_{\epsilon}^-$ to the pair of points $(-1, \text{triv}), (-1, \rho) \in (T // W)_2$.

By proposition 4.1 and the above argument, we may represent that part of the tempered dual $\mathbf{Irr}^{\text{temp}}(\text{SL}_2(\mathbb{F}_q((\varpi))))$ which corresponds to the unitary principal series in a diagram along the lines of [11, p.418].



The first double point represent the L -packet $\{\pi_{a_0}^+, \pi_{a_0}^-\}$. The second and third double-points represent, respectively, the L -packets $\{\pi_a^+, \pi_a^-\}$ and $\{\pi_{a_0+a}^+, \pi_{a_0+a}^-\}$. The second half-circle is repeated countably many times, and is parametrized by L -parameters $\{\varphi_a\}_{a \in \wp(K)}$, see theorem 4.4.

TOPOLOGY ON THE TEMPERED DUAL. Let $\mathcal{G} = \text{SL}_2(\mathbb{F}_q((\varpi)))$. The tempered dual of \mathcal{G} is the disjoint union $X = X_{\mathcal{G}}$ of the discrete series and the irreducible constituents in the principal series. We equip

X with the following topology \mathfrak{T} : The topology \mathfrak{T} must induce the standard topologies on each point, each copy of \mathbb{T} , and each copy (except one) of $(\mathbb{T}/W)_2$, *all of which (except one)* must become \mathfrak{T} -open sets. On the exceptional copy of $(\mathbb{T}/W)_2$ the Steinberg point St_G must be \mathfrak{T} -isolated. Then \mathfrak{T} is a locally compact topology on X . It is not Hausdorff.

In the space X , each L -packet in the unitary principal series will feature as a \mathfrak{T} -double-point.

There will be countably many double-points, one for each quadratic extension $K(\alpha)$; *cf.* the diagram in [11] for the tempered dual of $\text{SL}_2(\mathbb{Q}_p)$ with $p > 2$. In that diagram, there are just three double-points. For $\text{SL}_2(\mathbb{Q}_2)$ there would be seven double-points.

Each supercuspidal L -packet will feature as four \mathfrak{T} -isolated points in X .

We conjecture that \mathfrak{T} coincides with the Jacobson topology on the primitive ideal space of the reduced C^* -algebra of \mathcal{G} .

6. BIQUADRATIC EXTENSIONS OF $\mathbb{F}_q((\varpi))$

Quadratic extensions L/K are obtained by adjoining an \mathbb{F}_2 -line $D \subset K/\wp(K)$. Therefore, $L = K(\wp^{-1}(D)) = K(\alpha)$ where $D = \text{span}\{a + \wp(K)\}$, with $\alpha^2 - \alpha = a$. In particular, if a_0 is integer, the \mathbb{F}_2 -line $V_0 = \text{span}\{a_0 + \wp(K)\}$ contains all the cosets $a_i + \wp(K)$ where a_i is an integer and so $K(\wp^{-1}(\mathfrak{o})) = K(\wp^{-1}(V_0)) = K(\alpha_0)$ where $\alpha_0^2 - \alpha_0 = a_0$ gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering planes $W = \text{span}\{a + \wp(K), b + \wp(K)\} \subset K/\wp(K)$. Therefore, if $a + \wp(K)$ and $b + \wp(K)$ are \mathbb{F}_2 -linearly independent then $K(\wp^{-1}(W)) := K(\alpha, \beta)$ is biquadratic, where $\alpha^2 - \alpha = a$ and $\beta^2 - \beta = b$, $\alpha, \beta \in K^s$. Therefore, $K(\alpha, \beta)/K$ is biquadratic if $b - a \notin \wp(K)$.

A biquadratic extension containing the line V_0 is of the form $K(\alpha_0, \beta)/K$. There are countably many quadratic extensions L_0/K containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions L/K which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\alpha, \beta)/K$.

6.1. Ramification. The space $K/\wp(K)$ comes with a filtration

$$(6.1) \quad 0 \subset_1 V_0 \subset_f V_1 = V_2 \subset_f V_3 = V_4 \subset_f \dots \subset K/\wp(K)$$

where V_0 is the image of \mathfrak{o}_K and V_i ($i > 0$) is the image of \mathfrak{p}^{-i} under the canonical surjection $K \rightarrow K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset_f V_{2i+1}$ is a sub- \mathbb{F}_2 -space of codimension f . The

\mathbb{F}_2 -dimension of V_n is

$$(6.2) \quad \dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil,$$

where $\lceil x \rceil$ is the smallest integer not less than x .

Let L/K denote a Galois extension with Galois group G . For each $i \geq -1$ we define the i^{th} -ramification subgroup of G (in the lower numbering) to be:

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{p}_L^{i+1}, \forall x \in \mathfrak{o}_L\}.$$

An integer t is a *break* for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the Hasse-Herbrand function $\psi = \psi_{L/K}$:

$$G^u = G_{\psi(u)}.$$

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(0) = 0$.

Let $G_2 = \text{Gal}(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^\times/K^{\times 2}$ (Proposition 2.5), the pairing $K^\times/K^{\times 2} \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$ from (2.1) coincides with the pairing $G_2 \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$.

The profinite group G_2 comes equipped with a ramification filtration $(G_2^u)_{u \geq -1}$ in the upper numbering, see [5, p.409]. For $u \geq 0$, we have an orthogonal relation [5, Proposition 17]

$$(6.3) \quad (G_2^u)^\perp = \overline{\mathfrak{p}^{-\lceil u \rceil + 1}} = V_{\lceil u \rceil - 1}$$

under the pairing $G_2 \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will first compute the upper breaks and then use the Hasse-Herbrand function to compute the lower breaks in order to obtain the lower ramification filtration.

According to [5, Proposition 17], the positive breaks in the filtration $(G^v)_v$ occur precisely at integers prime to p . So, for $ch(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If G is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^v)_v$ (see [5], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane $W \subset K/\wp(K)$, the filtration (6.1) $(V_i)_i$ on $K/\wp(K)$ induces a filtration $(W_i)_i$ on W , where $W_i = W \cap V_i$. There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

Case 1 : W contains the line V_0 , i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of K . The extension has residue degree $f(L_0/K) = 2$ and ramification index $e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration $(W_i)_i$ looks like

$$0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.$$

By the orthogonality relation (6.3), the upper ramification filtration on $G = \text{Gal}(L_0/K)$ looks like

$$\{1\} = \dots = G^{t+1} \subset_1 G^t = \dots = G^0 \subset_1 G^{-1} = G$$

Therefore, the upper ramification breaks occur at -1 and t . The lower ramification breaks can be computed using the Hasse-Herbrand function. The table for the index of G^u in G^0 is as follows:

$u \in$	$[0, t]$	$]t, +\infty[$
$G^u =$	G^0	$\{1\}$
$(G^0 : G^u) =$	1	2

We have, $\psi(t) = \int_0^t (G^0 : G^u) du = t$, and the lower ramification breaks occur at -1 and t . It follows that the **lower filtration** is

$$(6.4) \quad G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_0 = \dots = G_t = \mathbb{Z}/2\mathbb{Z} ; G_{t+1} = \{1\}$$

The number of such W is equal to the number of planes in V_t containing the line V_0 but not contained in the subspace V_{t-1} . Note that this number can be computed and equals the number of biquadratic extensions of K containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1, t)$, $t > 0$ and odd.

Example 6.1. *The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks $(-1, 1)$ is equal to the number of planes in an $1 + f$ -dimensional \mathbb{F}_2 -space, containing the line V_0 . There are precisely*

$$1 + 2 + 2^2 + \dots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

of such biquadratic extensions.

Case 2.1 : W does not contain the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer t , necessarily odd.

The number of such W is equal to the number of planes in V_t whose intersection with V_{t-1} is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((\varpi))$, **case 2.1** does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification ramification filtration on $G = \text{Gal}(L/K)$ looks like

$$\{1\} = \dots = G^{t+1} \subset_2 G^t = \dots = G^{-1} = G$$

Therefore, there is a single upper ramification break occur at $t > 0$ and necessarily odd. The lower ramification breaks occurs at the same t , since we have:

$$\begin{array}{c} u \in \quad [0, t] \quad]t, +\infty[\\ \hline (G^0 : G^u) = \quad \begin{array}{ccc} G^0 & & \{1\} \\ 1 & & 2^2 \end{array} \end{array}$$

and so, $\psi(t) = \int_0^t (G^0 : G^u) du = t$, and the lower ramifications breaks occur at -1 and t . It follows that the **lower filtration** is

$$(6.5) \quad G_{-1} = G = \dots = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_{t+1} = \{1\}$$

For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes W contained in an \mathbb{F}_2 -space of dimension $1 + fi$, $t = 2i - 1$, which are transverse to a given codimension- f \mathbb{F}_2 -space.

Case 2.2 : W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers t_1 and t_2 , necessarily odd, with $0 < t_1 < t_2$.

The orthogonality relation for this case implies that the upper ramification filtration on $G = \text{Gal}(L/K)$ looks like

$$\{1\} = \dots = G^{t_2+1} \subset_1 G^{t_2} = \dots = G^{t_1+1} \subset_1 G^{t_1} = \dots = G$$

The upper ramification breaks occur at odd integers t_1 and t_2 .

Now, index of G^u in G^0 is:

$$\begin{array}{c} u \in \quad [0, t_1] \quad]t_1, t_2] \quad]t_2, +\infty[\\ \hline (G^0 : G^u) = \quad \begin{array}{ccc} 1 & 2 & 2^2 \end{array} \end{array}$$

The lower breaks occur at

$$\psi(t_1) = \int_0^{t_1} (G^0 : G^u) du = t_1.$$

and at

$$\begin{aligned}\psi(t_2) &= \int_0^{t_2} (G^0 : G^u) du = \int_0^{t_1} (G^0 : G^u) du + \int_{t_1}^{t_2} (G^0 : G^u) du \\ &= t_1 + 2(t_2 - t_1) = 2t_2 - t_1.\end{aligned}$$

In this case, the lower breaks occur at t_1 and $2t_2 - t_1$, with $0 < t_1 < t_2$ the odd integers where the upper ramification breaks occur.

We conclude that the **lower filtration** is given by

$$(6.6) \quad G = G_0 = \dots = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$(6.7) \quad G_{t_1+1} = \dots = G_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z} ; G_{2t_2-t_1+1} = \{1\}$$

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks (or lower breaks) (t_1, t_2) .

7. FORMAL DEGREES

In this section, we are influenced by the lecture notes of Reeder [12], and the preceding three talks in Washington, DC. For $\mathcal{G} = \mathrm{SL}_2(K)$, the dual group $G = \mathrm{SO}_3(\mathbb{C})$ contains a unique (up to conjugacy) subgroup $J \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, whose nontrivial elements are 180-degree rotations about three orthogonal axes. One can check that the centralizer and normalizer of J are given by

$$C_G(J) = J, \quad N_G(J) = O$$

where $O \simeq S_4$ is the rotation group of the octahedron whose vertices are the unit vectors on the given orthogonal axes. The quotient $O/J \simeq \mathrm{GL}_2(\mathbb{Z}/2)$ is the full automorphism group of J .

Each bi-quadratic extension L/K gives a surjective homomorphism

$$\varphi_L : \mathbf{W}_F \rightarrow J$$

which is a discrete parameter with $S_{\varphi_L} = J$, since $C_G(J) = J$, and whose conjugacy class depends only on L , since $O/J = \mathrm{Aut}(J)$.

Since

$$|S_{\varphi_L}| = 4$$

the L -packet Π_{φ_L} has 4 constituents. There are countably many bi-quadratic extensions, therefore there are countably many L -packets with 4 constituents.

None of these packets contains the Steinberg representation $\mathrm{St}_{\mathcal{G}}$ and so, according to Conjecture 6.1.4 in [12], these are all supercuspidal L -packets, each with 4 elements.

Consider the principal parameter:

$$\mathrm{Ad} \varphi_0 : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PSL}_2(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$$

The *adjoint gamma value* is given by

$$\gamma(\varphi_0) = \frac{q}{1 + q^{-1}}$$

where $q = 2^f$.

Concerning the adjoint gamma value $\gamma(\varphi)$ we have

$$\text{Ad } \varphi : \mathbf{W}_K \rightarrow J \rightarrow \text{SO}_3(\mathbb{C}) \rightarrow^{\text{Ad}} \text{Aut}(\mathfrak{so}_3(\mathbb{C}))$$

The adjoint representation of $\text{SO}_3(\mathbb{C})$ is equivalent to the standard representation of $\text{SO}_3(\mathbb{C})$ on \mathbb{C}^3 and so we replace the above sequence of maps by

$$\text{Ad } \varphi : \mathbf{W}_K \rightarrow J \rightarrow \text{SO}_3(\mathbb{C}).$$

For the L -function, we have

$$L(\text{Ad } \varphi, s) = \frac{1}{1 + q^{-s}}$$

and so we have

$$\gamma(\varphi) = \frac{2}{1 + q^{-1}} \cdot \varepsilon(\varphi)$$

where

$$\varepsilon(\varphi) = \pm q^{\alpha(\varphi)/2}.$$

Note that we have

$$(7.1) \quad \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = \frac{2}{q} \cdot \varepsilon(\varphi).$$

Now $\alpha(\varphi)$ is the Weil-Deligne version of the Artin conductor which is give here by

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]}$$

see [12], Reeder's notation.

We have to take the cases separately, beginning with (6.4).

Case 1: We have

$$G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_0 = \dots = G_t = \mathbb{Z}/2\mathbb{Z} ; G_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = (1 + t)2$$

According to Conjecture 6.1(1) in [14], we have

$$\text{Deg}(\pi_{\varphi_L}(\rho)) = \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| = 2^{t-f}$$

the *canonical formal degree* of each supercuspidal constituent in the packet Π_{φ_L} , i.e. the formal degree w.r.t. the Euler-Poincaré measure on \mathcal{G} . If we fix the field K , then the formal degree tends to ∞ as the break number t tends to ∞ .

The least allowed value of t is $t = 1$. When $t = f = 1$, the canonical formal degree of each element in the packet Π_{φ_L} is equal to 1. The lower ramification filtration is

$$G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_0 = G_1 = \mathbb{Z}/2\mathbb{Z} ; G_2 = \{1\}$$

and so, according to 6.1(5) in [12], the elements in this packet are not of depth zero.

Case 2.1: The lower ramification filtration is

$$G_{-1} = G = \dots = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_{t+1} = \{1\}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t+1)3$$

According to 6.1(1) in [12], we have

$$\begin{aligned} \text{Deg}(\pi_{\varphi_L}(\rho)) &= \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| \\ &= \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \\ &= \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \\ &= \frac{1}{2q} \cdot 2^{3(1+t)/2} \\ &= 2^{3(1+t)/2-f-1} \end{aligned}$$

Note that t is odd, therefore the formal degree is a *rational* number.

Case 2.2: This case admits the following lower ramification filtration:

$$\begin{aligned} G &= G_0 = \dots = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ G_{t_1+1} &= \dots = G_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z} ; G_{2t_2-t_1+1} = \{1\} \end{aligned}$$

We have

$$\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t_1+1)3 + \frac{(2t_2)2}{2} = 3 + 3t_1 + 2t_2$$

and, according to 6.1(1) in [12], we have

$$\begin{aligned}
 \text{Deg}(\pi_{\varphi_L}(\rho)) &= \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| \\
 &= \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \\
 &= \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \\
 &= \frac{1}{2q} \cdot 2^{3(1+t_1)/2+t_2} \\
 &= 2^{3(1+t_1)/2+t_2-f-1}
 \end{aligned}$$

the canonical formal degree of each supercuspidal in the packet Π_{φ_L} . If we fix f , then the formal degree tends to ∞ as the break numbers tend to ∞ .

Note that t_1 is odd, therefore all the formal degrees are *rational* numbers, in conformity with the rationality of the gamma ratio [7, Prop. 4.1].

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